Lösungen Differenzialrechnung II

Ausführliche Lösungen:

A1	Ausführliche Lösung	
	a)	$f(x) = x^2 + 3$ $x = 2: \frac{\Delta y}{\Delta x} = \frac{f(2+\Delta x) - f(2)}{\Delta x} = \frac{(2+\Delta x)^2 + 3 - (2^2+3)}{\Delta x}$ $= \frac{4 + 4\cdot\Delta x + (\Delta x)^2 + 3 - 4 - 3}{\Delta x} = \frac{4\cdot\Delta x + (\Delta x)^2}{\Delta x} = \frac{\cancel{\Delta x}\cdot(4+\Delta x)}{\cancel{\Delta x}} = 4 + \Delta x$ $\Rightarrow f'(2) = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}(4+\Delta x) = \underline{\underline{4}}$ $x = u: \frac{\Delta y}{\Delta x} = \frac{f(u+\Delta x) - f(u)}{\Delta x} = \frac{(u+\Delta x)^2 + 3 - (u^2+3)}{\Delta x}$ $= \frac{u^2 + 2u\cdot\Delta x + (\Delta x)^2 + 3 - u^2 - 3}{\Delta x} = \frac{2u\cdot\Delta x + (\Delta x)^2}{\Delta x} = \frac{\cancel{\Delta x}\cdot(2u+\Delta x)}{\cancel{\Delta x}} = 2u + \Delta x$ $\Rightarrow f'(u) = \lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}(2u+\Delta x) = \underline{\underline{2u}}$

A1	Ausführliche Lösung

b)

$$f(x) = \frac{2}{x}$$

$$x = 2: \frac{\Delta y}{\Delta x} = \frac{f(2+\Delta x) - f(2)}{\Delta x} = \frac{\frac{2}{2+\Delta x} - \frac{2}{2}}{\Delta x} = \frac{\frac{2}{2+\Delta x} - 1}{\Delta x} = \frac{\frac{2}{2+\Delta x} - \frac{1\cdot(2+\Delta x)}{2+\Delta x}}{\Delta x}$$

$$= \frac{\frac{2-1\cdot(2+\Delta x)}{2+\Delta x}}{\Delta x} = \frac{\frac{2-2-\Delta x}{2+\Delta x}}{\Delta x} = \frac{\frac{-\Delta x}{2+\Delta x}}{\Delta x} = \frac{\frac{-\Delta x}{2+\Delta x}}{\frac{\Delta x}{1}} = \frac{-1\cdot \cancel{\Delta x}}{\cancel{\Delta x}(2+\Delta x)} = \frac{-1}{2+\Delta x}$$

$$f'(2) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{-1}{2+\Delta x}\right) = \underline{\underline{-\frac{1}{2}}}$$

$$x = u: \frac{\Delta y}{\Delta x} = \frac{f(u+\Delta x) - f(2)}{\Delta x} = \frac{\frac{2}{u+\Delta x} - \frac{2}{u}}{\Delta x} = \frac{\frac{2\cdot u}{u\cdot(u+\Delta x)} - \frac{2\cdot(u+\Delta x)}{u\cdot(u+\Delta x)}}{\Delta x}$$

$$= \frac{\frac{2\cdot u - 2\cdot(u+\Delta x)}{u\cdot(u+\Delta x)}}{\Delta x} = \frac{\frac{2\cdot u - 2\cdot u - 2\cdot\Delta x}{u\cdot(u+\Delta x)}}{\Delta x} = \frac{\frac{-2\cdot\Delta x}{u\cdot(u+\Delta x)}}{\frac{\Delta x}{1}}$$

$$= \frac{-2\cdot\cancel{\Delta x}}{\cancel{\Delta x}\cdot u\cdot(u+\Delta x)} = \frac{-2}{u\cdot(u+\Delta x)} = \frac{-2}{u^2 + u\cdot\Delta x}$$

$$f'(u) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\left(\frac{-2}{u^2 + u\cdot\Delta x}\right) = \underline{\underline{-\frac{2}{u^2}}}$$

A1 Ausführliche Lösung

c)

$$f(x) = \frac{1}{x+1}$$

$$x = 2: \frac{\Delta y}{\Delta x} = \frac{f(2+\Delta x) - f(2)}{\Delta x} = \frac{\frac{1}{2+\Delta x+1} - \frac{1}{2+1}}{\Delta x}$$

$$= \frac{\frac{1}{3+\Delta x} - \frac{1}{3}}{\Delta x} = \frac{\frac{1 \cdot 3}{3 \cdot (3+\Delta x)} - \frac{1 \cdot (3+\Delta x)}{3 \cdot (3+\Delta x)}}{\Delta x}$$

$$= \frac{\frac{1 \cdot 3 - 1 \cdot (3+\Delta x)}{3 \cdot (3+\Delta x)}}{\Delta x} = \frac{\frac{3 - 3 - \Delta x}{3 \cdot (3+\Delta x)}}{\Delta x} = \frac{\frac{-\Delta x}{3 \cdot (3+\Delta x)}}{\frac{\Delta x}{1}} = \frac{-1 \cdot \cancel{\Delta x}}{\cancel{\Delta x} \cdot 3 \cdot (3+\Delta x)} = \frac{-1}{3 \cdot (3+\Delta x)}$$

$$f'(2) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \left(\frac{-1}{3(3+\Delta x)} \right) = \underline{\underline{-\frac{1}{9}}}$$

$$x = u: \frac{\Delta y}{\Delta x} = \frac{f(u+\Delta x) - f(2)}{\Delta x} = \frac{\frac{1}{u+\Delta x+1} - \frac{1}{u+1}}{\Delta x}$$

$$= \frac{\frac{1 \cdot (u+1)}{(u+1) \cdot (u+\Delta x+1)} - \frac{1 \cdot (u+\Delta x+1)}{(u+1) \cdot (u+\Delta x+1)}}{\Delta x}$$

$$= \frac{\frac{1 \cdot (u+1) - 1 \cdot (u+\Delta x+1)}{(u+1) \cdot (u+\Delta x+1)}}{\Delta x} = \frac{\frac{u+1-u-\Delta x-1}{(u+1) \cdot (u+\Delta x+1)}}{\Delta x} = \frac{\frac{-\Delta x}{(u+1) \cdot (u+\Delta x+1)}}{\frac{\Delta x}{1}}$$

$$= \frac{-1 \cdot \cancel{\Delta x}}{\cancel{\Delta x} \cdot (u+1) \cdot (u+\Delta x+1)} = \frac{-1}{(u+\Delta x+1)(u+1)}$$

$$f'(u) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{-1}{(u+\Delta x+1)(u+1)} = \frac{-1}{(u+1)(u+1)} = \underline{\underline{-\frac{1}{(u+1)^2}}}$$

A1 Ausführliche Lösung

d) $f(x)=\sqrt{x}$

$$x=2:\ \frac{\Delta y}{\Delta x}=\frac{f(2+\Delta x)-f(2)}{\Delta x}=\frac{\sqrt{2+\Delta x}-\sqrt{2}}{\Delta x}=\frac{\left(\sqrt{2+\Delta x}-\sqrt{2}\right)\cdot\left(\sqrt{2+\Delta x}+\sqrt{2}\right)}{\Delta x\cdot\left(\sqrt{2+\Delta x}+\sqrt{2}\right)}$$

$$=\frac{2+\Delta x-2}{\Delta x\cdot\left(\sqrt{2+\Delta x}+\sqrt{2}\right)}=\frac{1\cdot\cancel{\Delta x}}{\cancel{\Delta x}\cdot\left(\sqrt{2+\Delta x}+\sqrt{2}\right)}=\frac{1}{\sqrt{2+\Delta x}+\sqrt{2}}$$

$$f'(2)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{1}{\sqrt{2+\Delta x}+\sqrt{2}}\right)=\frac{1}{\sqrt{2}+\sqrt{2}}=\underline{\underline{\frac{1}{2\sqrt{2}}}}$$

$$x=u:\ \frac{\Delta y}{\Delta x}=\frac{f(u+\Delta x)-f(u)}{\Delta x}=\frac{\sqrt{u+\Delta x}-\sqrt{u}}{\Delta x}=\frac{\left(\sqrt{u+\Delta x}-\sqrt{u}\right)\cdot\left(\sqrt{u+\Delta x}+\sqrt{u}\right)}{\Delta x\cdot\left(\sqrt{u+\Delta x}+\sqrt{u}\right)}$$

$$=\frac{u+\Delta x-u}{\Delta x\cdot\left(\sqrt{u+\Delta x}+\sqrt{u}\right)}=\frac{1\cdot\cancel{\Delta x}}{\cancel{\Delta x}\cdot\left(\sqrt{u+\Delta x}+\sqrt{u}\right)}=\frac{1}{\sqrt{u+\Delta x}+\sqrt{u}}$$

$$f'(u)=\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}=\lim_{\Delta x\to 0}\left(\frac{1}{\sqrt{u+\Delta x}+\sqrt{u}}\right)=\frac{1}{\sqrt{u}+\sqrt{u}}=\underline{\underline{\frac{1}{2\sqrt{u}}}}$$

A2 Ausführliche Lösung

a) $f(x)=-2x^4+3x^2-4x+2\Rightarrow f'(x)=-8x^3+6x-4$

b) $f(x)=0{,}5x^4-x^3+2{,}5x^2-8\Rightarrow f'(x)=2x^3-3x^2+5x$

c) $f(x)=\frac{1}{32}x^3+\frac{3}{2}x-4\Rightarrow f'(x)=\frac{3}{32}x^2+\frac{3}{2}$

d) $s(t)=-\frac{5}{6}t^2+\frac{2}{3}t+\frac{5}{2}\Rightarrow s'(t)=-\frac{5}{3}t+\frac{2}{3}$

e) $f(x)=-(x-6)^2(x+1)=-x^3+11x^2-24x-36\Rightarrow f'(x)=-3x^2+22x-24$

f) $f(x)=\frac{1}{2}\left(x^2-2\right)^2=\frac{1}{2}x^4-2x^2+2\Rightarrow f'(x)=2x^3-4x$

A3 Ausführliche Lösung

a) $f(x)=\frac{1}{16}\left(x^3+x-1\right)\Rightarrow f'(x)=\frac{3}{16}x^2+\frac{1}{16}$

b) $f(x)=x\left(x^2-\frac{3}{2}x-4\right)=x^3-\frac{3}{2}x^2-4x\Rightarrow f'(x)=3x^2-3x-4$

c) $f(x)=ax^4+bx^2+c\Rightarrow f'(x)=4ax^3+2bx$

d) $f(x)=ax^3+bx^2+cx+d\Rightarrow f'(x)=3ax^2+2bx+c$

e) $f(x)=6x+\frac{5}{x}=6x+5x^{-1}\Rightarrow f'(x)=6-5x^{-2}=6-\frac{5}{x^2}$

f) $f(x)=x^3-2x^2+\frac{1}{x}=x^3-2x^2+x^{-1}\Rightarrow f'(x)=3x^2-4x-x^{-2}=3x^2-4x-\frac{1}{x^2}$

A4	Ausführliche Lösung	
	a)	$f_t(x)=\frac{t}{2}x^4-2tx^3+t^2 \Rightarrow f_t'(x)=2tx^3-6tx^2$
	b)	$f_k(x)=\frac{1}{k}x^3+kx^2+(k+1)x \Rightarrow f'_k(x)=\frac{3}{k}x^2+2kx+k+1$
	c)	$f_a(x)=\frac{1}{4}x^3+ax^2+\left(a-\frac{1}{2}\right)x-3 \Rightarrow f'_a(x)=\frac{3}{4}x^2+2ax+a-\frac{1}{2}$
	d)	$f_t(x)=\frac{1}{2t}\left(x^2-t\right)^2=\frac{1}{2t}x^4-x^2+\frac{1}{2}t \Rightarrow f_t'(x)=\frac{2}{t}x^3-2x$
	e)	$f(t)=5t^3-2t+5 \Rightarrow f'(t)=15t^2-2$
	f)	$f(z)=-1{,}5z^3+2{,}5z^2+z \Rightarrow f'(z)=-4{,}5z^2+5z+1$
	g)	$A(u)=\frac{1}{2}u^2+3u+2u+1 \Rightarrow A'(u)=u+5$
	h)	$A(u)=\frac{1}{2}u\left(u^2+1\right)=\frac{1}{2}u^3+\frac{1}{2}u \Rightarrow A'(u)=\frac{3}{2}u^2+\frac{1}{2}$

A5	Ausführliche Lösung	
	a)	$f(x)=3x^2-5 \Rightarrow f'(x)=6x \Rightarrow f'(-3)=6\cdot(-3)=\underline{\underline{-18}}$ Nullstellen von f(x): $f(x)=0 \Leftrightarrow 3x^2-5=0 \Rightarrow x_{1/2}=\pm\sqrt{\frac{5}{3}} \Rightarrow f'\left(\pm\sqrt{\frac{5}{3}}\right)=6\cdot\left(\pm\sqrt{\frac{5}{3}}\right)=\underline{\underline{\pm 6\cdot\sqrt{\frac{5}{3}}}}$

A5	Ausführliche Lösung	
	b)	$f(x)=4x-\frac{1}{x} \Rightarrow f'(x)=4+\frac{1}{x^2} \Rightarrow f'(-3)=4+\frac{1}{9}=\underline{\underline{\frac{37}{9}}}$ Nullstellen von f(x): $f(x)=0 \Leftrightarrow 4x-\frac{1}{x}=0 \Leftrightarrow \frac{4x^2-1}{x}=0 \Leftrightarrow x^2=\frac{1}{4} \Rightarrow x_{1/2}=\pm\frac{1}{2}$ $\Rightarrow f'\left(\pm\frac{1}{2}\right)=4+\frac{1}{\left(\pm\frac{1}{2}\right)^2}=4+\frac{1}{\frac{1}{4}}=4+4=\underline{\underline{8}}$

A6	Ausführliche Lösungen	
	a)	$f(x) = 2x^2 + 3x + 1 \Rightarrow f'(x) = 4x + 3$
	b)	$f(x) = -x^2 + 2x - 1 \Rightarrow f'(\ ) = -2x + 2$
	c)	$f(x) = x + 1 \Rightarrow f'(x) = 1$
	d)	$f(x) = \frac{1}{2}x^2 - \frac{1}{3}x + 4 \Rightarrow f'(x) = x - \frac{1}{3}$
	e)	$f(x) = 5b\,;\, b \in \mathbb{R} \setminus \{0\} \Rightarrow f'(x) = 0$
	f)	$f(x) = \frac{3}{4}x^2 - \frac{2}{3}x - 1 \Rightarrow f'(x) = \frac{3}{2}x - \frac{2}{3}$
	g)	$f(x) = 2x^3 + 3x^2 - 4x + 2 \Rightarrow f'(x) = 6x^2 + 6x - 4$
	h)	$f(\ ) = -\frac{1}{4}x^3 + \frac{2}{3}x^2 - \frac{3}{4}x + \frac{1}{2} \Rightarrow f'(x) = -\frac{3}{4}x^2 + \frac{4}{3}x - \frac{3}{4}$
	i)	$f(x) = \frac{1}{2}x^2 + 3x - 7 \Rightarrow f'(x) = x + 3$
	j)	$f(x) = \frac{3}{4}x^2 + 5x + 8 \Rightarrow f'(x) = \frac{3}{2}x + 5$